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Squeezed State Projectors in String Field Theory

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Abstract

We find a new subalgebra of the star product in the matter sector. Its elements are squeezed states whose matrices commute with $(\mathrm{K}_1)^2$. This subalgebra contains a large set of projectors. The states are represented by their eigenvalues and we find a mapping between the eigenvalues representation and other known representations. The sliver is naturally in this subalgebra. Surprisingly, all the generalized butterfly states are also in this subalgebra, enabling us to analyze their spectrum, and to show the orthogonality of different butterfly states. This means that multi D-brane states can be built of butterfly states.

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1 Introduction and summary

Cubic string field theory [?] has attracted renewed interest, mainly due to Sen's conjecture on tachyon condensation [?]. For bosonic string theory the conjecture is that the open string tachyon potential has a minimum that the tachyon condenses to. The value of the potential at this minimum is exactly equal to the tension of a D-brane. The tachyon potential was calculated numerically in cubic open string field theory using the level truncation scheme [?, ?, ?, ?, ?]. These calculations agree very well with Sen's conjecture.

Kostelecky and Potting attempted in [?] to find the vacuum of string field theory analytically. The basis for their solution in the matter sector was a solution to the projection equation

$$\mathcal{S} = \mathcal{S} \star \mathcal{S} \,, \tag{1.1}$$

where the product is the star product of cubic string field theory [?]. They assumed that the solution is a squeezed state of the form

$$|\mathcal{S}\rangle = \exp(-\frac{1}{2}\sum_{n,m=1}^{\infty} a_n^\dagger S_{nm} a_m^\dagger)\,|0\rangle \ . \tag{1.2}$$

They made one more simplifying ansatz

$$[T, V^{rs}] = 0 \,, \qquad {\scriptstyle r,s=1\ldots 3} \,, \tag{1.3}$$

where $T = CS$, the twist matrix $C_{nm} = (-1)^m \delta_{nm}$ and V^{rs} are the three-vertex matrices [?]. These assumptions allow for only two solutions. One trivial solution which is the identity state $S = C$ and one non-trivial solution. A little earlier the subalgebra of wedge states was found [?]. Two of those states are projectors: the 360° wedge, which corresponds to the identity state and the infinitely thin wedge called the sliver [?], which corresponds to the non-trivial solution of [?].

It seems surprising that the first two non-trivial projection states found, turned out to be the same state, especially due to the fact that in [?] CFT techniques were used while [?] used oscillators. The reason oscillator based calculations gave a wedge state is that the three-vertex matrices are related to the 120° wedge state and all wedge state matrices commute. Therefore the ansatz (1.3) restricts the solutions to wedge state solutions.

Projectors are even more relevant in vacuum string field theory (VSFT), which is a formulation of SFT around the tachyon vacuum [?, ?, ?, ?]. In this formulation the kinetic operator $\mathcal{Q}$ is purely ghost, meaning that the equation of motion for the matter sector is simply the projection equation (1.1). Therefore, projectors are solitonic solutions of VSFT associated with D-branes. The expectation that there will be only one type of $D25$-brane does not agree with the infinite number of spatially independent rank-one projectors. This suggests that all these projectors are related by a gauge transformation.

Rastelli Sen and Zwiebach found the spectrum of the three-vertex and wedge states matrices [?]. First the eigenvalues and eigenvectors of the matrix K_1 were calculated, where the matrix K_1 is defined as the action of the star algebra derivation $K_1 = L_1 + L_{-1}$ in the oscillator basis. The eigenvalues of K_1 are continuous in the range $-\infty < \kappa < \infty$ and are non-degenerate. K_1 also satisfies the commutation relations

$$[M^{rs}, \mathrm{K}_1] = [T_N, \mathrm{K}_1] = 0\,, \qquad {\scriptstyle r,s=1\ldots 3}\,, \quad {\scriptstyle N=1\ldots\infty}\,, \tag{1.4}$$

where $V^{rs} = CM^{rs}$ are the three-vertex matrices and $V_N = CT_N$ are the wedge state matrices. The eigenvectors of K_1 are the eigenvectors of M^{rs}, T_N because K_1 is non-degenerate and eq. (1.4). The spectroscopy results simplified many elaborate computations. It was used in [?] to formulate the continuous Moyal representation of the star algebra, in [?] to study the gauge transformation of the vector state in VSFT, in [?] for an analytical calculation of tensions ratio, and in [?] for proving the equivalence of two definitions of $\mathcal{Q}$.

The spectroscopy simplifies the calculations of [?] as well. We can work in the K_1 basis where the M^{rs} matrices are diagonal, and require that $T = CS$ should also commute with K_1

$$[T, \mathrm{K}_1] = 0\,. \tag{1.5}$$

Instead of equations involving infinite matrices, we get scalar equations for each eigenvalue κ. The condition (1.5) is exactly analog to the ansatz (1.3). Thus, repeating the calculations of [?], using the spectroscopy results, gives the same solutions.

The placement of the factors of C in the above commutation relations is very rigid due to the fact that C does not commute with K_1. The main idea of this paper is to rely on the commutation relation

$$[C, (\mathrm{K}_1)^2] = 0\,. \tag{1.6}$$

This is a result of the double degeneracy of $(\mathrm{K}_1)^2$ where each eigenvalue κ^2 has two eigenvectors $v^{(\pm\kappa)}$. We solve the equations of [?] using a weaker ansatz

$$[S,(\mathrm{K}_1)^2]=0\,. \tag{1.7}$$

States that satisfy this ansatz form a subalgebra $\mathcal{H}_{\kappa^2}$ of the star product. States in this subalgebra are block diagonal in the K_1 basis and are represented using two by two matrices. The advantage of the weaker ansatz is that now we get a larger set of solutions. The calculations and the solutions are presented in section 2.

Understanding the meaning of our solutions requires translating them into more familiar representations. This is done in section 3. It turns out that the entire family of generalized butterfly states [?, ?, ?] is in the $\mathcal{H}_{\kappa^2}$ subalgebra and we find their spectra (3.37).

The fact that the butterfly state

$$\exp(-\frac{1}{2}L_{-2})\,|0\rangle=\exp(-\frac{1}{2}a^\dagger V^B a^\dagger)\,|0\rangle\ , \tag{1.8}$$

satisfies our ansatz, meaning $[V_B,(\mathrm{K}_1)^2]=0$, was somewhat unexpected. An explicit calculation of the butterfly spectroscopy can be found in the appendix. It seems that other projectors with a simple Virasoro structure, do not satisfy our ansatz.

Not all our solutions correspond to surface states. One interesting such solution is the dual of the nothing state

$$|\mathcal{S}\rangle=\exp(-\frac{1}{2}a^\dagger(-I)a^\dagger)\,. \tag{1.9}$$

It looks like the nothing state only with an opposite sign in the exponent. The nothing state describes a configuration of a string with an X_n independent wave function, that is $\Phi[\pi_k]=\prod_{n=1}^\infty\delta(\pi_n)$, where π_n are the conjugate momenta. The dual of the nothing is $\Phi[X_k]=\prod_{n=1}^\infty\delta(X_n)$.

In section 3.5 we discuss the $\mathcal{H}_{\kappa^2}$ subalgebra in the Moyal representation. We show that all our projectors have the same normalization in this formalism and show the symmetry that relates them. We also show that the generalized butterfly states are orthogonal. This result can be used to build multi D-brane states from squeezed states only, without the need of non Gaussian states.

In the rest of the paper states are written up to normalization. The singular normalization of string field states is supposed to be corrected by the ghost sector, which we did not treat in this paper. We also ignored the zero modes, assuming that the string field is independent of them, meaning that our solutions are analog to $D25$-branes. The spacetime index $\mu = 0 \ldots 25$, is also suppressed. One can use the spectroscopy of the matter sector including the zero modes, and that of the ghost sector [?, ?, ?, ?], to generalize this work.

2 Projectors in the $\mathcal{H}_{\kappa^2}$ subalgebra

Squeezed states whose matrices commute with $(\mathrm{K}_1)^2$ form a subalgebra of the star product, which we denote $\mathcal{H}_{\kappa^2}$. To prove this statement we write the expression of the star product of two squeezed states, $|\mathcal{S}_3\rangle = |\mathcal{S}_1\rangle \star |\mathcal{S}_2\rangle$ using [?]

$$CS_3C = V^{11} + \begin{pmatrix} V^{12}, & V^{21} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1} - S_1 V^{11} & -S_1 V^{12} \\ -S_2 V^{21} & \mathbf{1} - S_2 V^{11} \end{pmatrix}^{-1} \cdot \begin{pmatrix} S_1 V^{21} \\ S_2 V^{12} \end{pmatrix}, \tag{2.1}$$

where V^{rs} are the three-vertex matrices. In [?] it was shown that

$$[M^{rs}, \mathrm{K}_1] = 0\,, \tag{2.2}$$

where $M^{rs} = CV^{rs}$. The fact that $[C, (\mathrm{K}_1)^2] = 0$ completes the proof, since S_3 is a function of matrices that commute with $(\mathrm{K}_1)^2$, and therefore $[S_3, (\mathrm{K}_1)^2] = 0$.

Eq. (2.2) with the nondegeneracy of K_1 implies that the M^{rs} matrices are diagonal in the K_1 basis. In [?] their eigenvalues were found

$$\begin{aligned} \mu(\kappa) \equiv \mu^{11}(\kappa) &= -\frac{1}{1 + 2\cosh(\kappa\pi/2)}\,, \\ \mu^{12}(\kappa) &= \frac{1 + \exp(\kappa\pi/2)}{1 + 2\cosh(\kappa\pi/2)}\,, \\ \mu^{21}(\kappa) &= \mu^{12}(-\kappa)\,. \end{aligned} \tag{2.3}$$

To find squeezed state projectors one has to solve eq. (2.1), setting $S_1 = S_2 = S_3$. In the $\mathcal{H}_{\kappa^2}$ subalgebra the projector condition becomes a set of equations consisting of one scalar equation for $\kappa = 0$ and one 2×2 matrix equation for each $\kappa > 0$. We shall now solve this set of equations for all κ to find the condition for a state in the $\mathcal{H}_{\kappa^2}$ subalgebra to be a projector.

2.1 The $\kappa = 0$ subspace

For the $\kappa = 0$ eigenvalue there is a single normalizable eigenvector of $(\mathrm{K}_1)^2$. We use the fact that it is twist odd to set $V^{rs} = -\mu^{rs}(0)$ in (2.1) and get

$$s_0 = \frac{1}{3} + \begin{pmatrix} -\frac{2}{3}, & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 - \frac{1}{3}s_0 & \frac{2}{3}s_0 \\ \frac{2}{3}s_0 & 1 - \frac{1}{3}s_0 \end{pmatrix}^{-1} \begin{pmatrix} s_0 & 0 \\ 0 & s_0 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} . \tag{2.4}$$

The matrix that has to be inverted has an inverse for $s_0 \neq 1, -3$. The condition $s_0 \neq 1$ is related to the fact that S defines a Bogoliubov transformation, and thus the eigenvalues of $S^\dagger S$ should obey

$$\lambda_{S^\dagger S} < 1 . \tag{2.5}$$

We know, however, that projectors of SFT are singular. When the matrix is inverted eq. (2.4) gives

$$s_0 = \pm 1 , \tag{2.6}$$

We see that although singular, $s_0 = 1$ is a solution to the equation. Indeed this is the solution for all surface state projectors with $f(\pm i) = \infty$, where $f(z)$ is the canonical transformation defining the state, as was shown in [**?**]. The $s_0 = -1$ solution can represent either a non-surface-state projector, or a surface state for which $f(\pm i) \neq \infty$, as in the case of the star-algebra identity $S = C$, for which

$$f(z) = \frac{z}{1 - z^2} . \tag{2.7}$$

2.2 The $\kappa \neq 0$ subspace

The eigenvalue κ^2 of $(\mathrm{K}_1)^2$ is doubly degenerate for $\kappa \neq 0$ with the eigenvectors $v^{(\pm\kappa)}$. To solve the projection equation for the $\pm\kappa$ pairs we need to work in the two dimensional subspace spanned by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv v^{(-\kappa)} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv v^{(\kappa)} . \tag{2.8}$$

We avoid a double counting by taking only $\kappa > 0$. In this subspace the entities of eq. (2.1) are two by two matrices. The action of the C matrix in this subspace is

$$C_\kappa = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{2.9}$$

since the vectors $v_\pm^{(\kappa)} = \frac{1}{2}\left(v^{(-\kappa)} \mp v^{(\kappa)}\right)$ [?] are eigenvectors of C with eigenvalues ± 1. Using eq. (2.3) we can now write the 3-vertex in this subspace

$$\begin{aligned} V^{11} &= \frac{1}{1+2\cosh(\kappa\pi/2)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ V^{12} &= \frac{-1}{1+2\cosh(-\kappa\pi/2)} \begin{pmatrix} 0 & 1+\exp(\kappa\pi/2) \\ 1+\exp(\kappa\pi/2) & 0 \end{pmatrix}, \\ V^{21} &= (V^{12})^T . \end{aligned} \tag{2.10}$$

In this subspace a state in the $\mathcal{H}_{\kappa^2}$ subalgebra is given by

$$S_\kappa = \begin{pmatrix} s_1(\kappa) & s_2(\kappa) \\ s_2(\kappa) & s_3(\kappa) \end{pmatrix}. \tag{2.11}$$

S_κ is symmetric because it represents a quadratic form.

The functions s_1, s_2, s_3 are complex in general, but we should impose the BPZ reality condition on the string field state. The condition is that the BPZ conjugate of the state should be equal to the hermitian conjugate of the state [?, ?, ?]

$$\langle V_2 | \Psi \rangle \equiv \langle \Psi |_{\mathrm{BPZ}} = (|\Psi\rangle)^\dagger . \tag{2.12}$$

For squeezed states the reality condition translates into the condition

$$CSC = S^\dagger , \tag{2.13}$$

and for the $\mathcal{H}_{\kappa^2}$ subalgebra we get the conditions

$$\begin{gathered} s_0 = s_0^* , \\ \begin{pmatrix} s_3 & s_2 \\ s_2 & s_1 \end{pmatrix} = \begin{pmatrix} s_1^* & s_2^* \\ s_2^* & s_3^* \end{pmatrix} \Rightarrow \begin{cases} s_3 = s_1^* \\ s_2 = s_2^* \end{cases} . \end{gathered} \tag{2.14}$$

Solving the projection equation we get two valid solutions (before imposing any reality condition)

$$s_1 = s_3 = 0, \quad s_2 = -1\,. \tag{2.15}$$

$$s_1 s_3 = s_2^2 - 2\cosh\left(\frac{\kappa\pi}{2}\right) s_2 + 1\,. \tag{2.16}$$

The first solution is $S_\kappa = C_\kappa$. If we take this solution for all values of $\kappa \neq 0$ and combine it with the $s_0 = -1$ solution of $\kappa = 0$, we get $S = C$, which is the star-algebra identity (2.7). The second solution is actually a two-parameter family of solutions. But not all solutions are legitimate. Being a Bogoliubov transformation we have the restriction (2.5) on the matrix S_κ. We shall allow for singular transformations,

$$\lambda_{S_\kappa^\dagger S_\kappa} = (s_2 \pm |s_1|)^2 \leq 1\,, \tag{2.17}$$

as we were forced to do in the $\kappa = 0$ case. We parameterize the solutions using the two invariants

$$\begin{aligned} u &\equiv \frac{\operatorname{tr} S_\kappa}{2} = \frac{s_1 + s_3}{2}\,, \\ v &\equiv -\det S_\kappa = 2\cosh(\frac{\kappa\pi}{2}) s_2 - 1\,, \end{aligned} \tag{2.18}$$

with the inverse relations

$$\begin{aligned} s_2 &= \frac{v+1}{2\cosh(\frac{\kappa\pi}{2})}\,, \\ s_{1,3} &= u \pm i\sqrt{\left(\frac{v+1}{2\cosh(\frac{\kappa\pi}{2})}\right)^2 - v - u^2}\,, \end{aligned} \tag{2.19}$$

where s_1 gets the plus sign and s_3 gets the minus sign, or vice versa. The BPZ reality condition implies that u, v are real as well as the square root in (2.19).

The condition for the solution to be twist invariant is

$$[S, C] = 0 \Rightarrow s_1 = s_3\,. \tag{2.20}$$

When combined with the BPZ reality condition (2.14), twist invariance enforces the reality of S_κ. In the u, v language this condition reads

$$u^2 = \left(\frac{v+1}{2\cosh(\frac{\kappa\pi}{2})}\right)^2 - v\,. \tag{2.21}$$

We will show that this solution contains the generalized butterfly projectors, including the sliver and the butterfly.

The projection equation (2.16) together with the normalization requirement (2.17) and the reality condition on S_κ (2.14) restricts u, v to the region

$$\begin{aligned} v &\geq -1\,, \\ \left(\frac{v+1}{2\cosh(\frac{\kappa\pi}{2})}\right)^2 - v &\geq u^2\,. \end{aligned} \tag{2.22}$$

These equations determine the allowed range for the solutions as a function of κ as illustrated in figure 1. To build a projector one has to choose an allowed value for (u, v) for every value of $\kappa > 0$. This prescription allows for a large class of projectors.

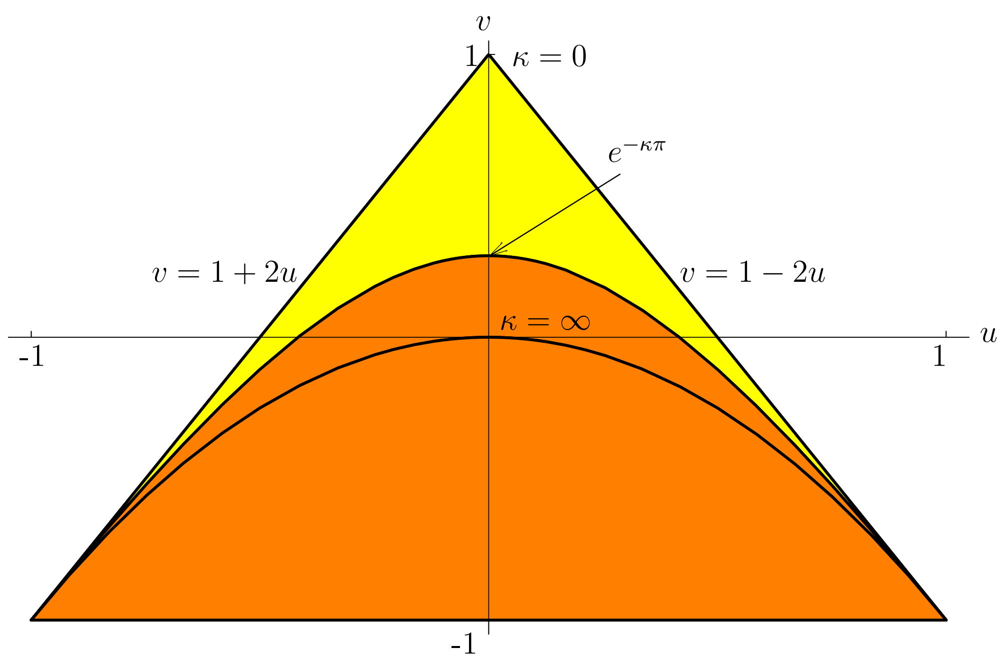

Figure 1: The allowed range for a projector in the u, v plane as a function of κ. The triangle represents the normalization condition (2.17). To satisfy the BPZ reality condition (2.14) the projector has to be below the limiting curve. For each value of $0 < \kappa < \infty$ there is a different limiting curve, cutting the v-axis at $e^{-\kappa\pi}$. Projectors on the curve are twist invariant (2.20).

u, v should be integrable functions of κ because the continuous κ basis is related to the oscillator basis by integration. Moreover, we should identify functions which differ on a zero measure set. Further restrictions on the functions u, v would be related to identifying which string fields are legitimate. What is the class of legitimate string fields is still unknown. It should be large enough to contain D-branes [**?**, **?**], perturbative states around D-branes [**?**], as well as closed strings [**?**]. Yet, it should not be too large [**?**, **?**]. Continuity of u, v as a functions of κ restricts the class of string fields. We do not know if it has anything to do with the "correct" choice, but we shall henceforth mention some of its consequences. In fact without this restriction the analysis of $\kappa = 0$ in section 2.1 is meaningless.

Continuity of $s_{1,2,3}$ at $\kappa = 0$ implies that one of the $\kappa = 0$ solutions (2.6) is reached in the $\kappa \to 0$ limit. The eigenvector $v^{(\kappa=0)}$ is twist odd[**?**]. Therefore, to get the relevant solution of S_κ we use the projector on the twist odd eigenvalue

$$P_\kappa^{(-)} = -\frac{1}{2}(C_\kappa - \mathbf{1}) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad (2.23)$$

In the limit $\kappa \to 0$ we get

$$\begin{aligned} S_\kappa^{(-)} = P_\kappa^{(-)} S_\kappa P_\kappa^{(-)} &\xrightarrow{\kappa \to 0} \left(u + \frac{v+1}{2}\right) \cdot P_\kappa^{(-)} = s_0 \cdot P_\kappa^{(-)} \\ &\Rightarrow \begin{cases} v = 1 - 2u & s_0 = 1 \\ v = -3 - 2u & s_0 = -1 \end{cases}. \end{aligned} \qquad (2.24)$$

We see that the SFT projectors (2.16) which are continuous with respect to κ will end either on the right segment of the figure and have $s_0 = 1$, or at the point $u = v = -1$, and have $s_0 = -1$. The identity (2.15) has $s_0 = -1$, since $P_\kappa^{(-)} C_\kappa P_\kappa^{(-)} = -1 \cdot P_\kappa^{(-)}$.

All the projectors in (2.16) are of rank one. This follows from the fact that a projector is given by a trajectory in the κ, u, v space. The rank can be given by a continuous function of this trajectory, all the trajectories are homotopic, and the previously known ones among them are of rank one. This observation is in accordance with the proof of [**?**] that all Gaussian projectors apart of the identity are of rank one. Of course both our argument, and that of [**?**] may fail if the given projector is too singular. Another property of rank one projectors is their factorization to functions of the left and right part of the string. We show that the projectors in (2.16) factorize in [**?**].

2.3 The sliver, the butterfly and the nothing

Among our solutions we can recognize the sliver, the butterfly and the nothing. We shall see in section 3.3.1 that in fact all the generalized butterfly states are in $\mathcal{H}_{\kappa^2}$. The spectrum of the sliver is given in [**?**]

$$\tau(\kappa) = -e^{-\frac{|\kappa|\pi}{2}}\,, \tag{2.25}$$

or in our notation

$$T_\kappa = C_\kappa S_\kappa = -\exp\left(-\frac{\kappa\pi}{2}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \Rightarrow S_\kappa = \exp\left(-\frac{\kappa\pi}{2}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\,. \tag{2.26}$$

Using eq. (2.18) we get for the sliver $u = 0$, $v = \exp(-\kappa\pi)$.

For the butterfly the eigenvalues in the $C^{(+)}$ subspace are zero, and in the $C^{(-)}$ subspace the eigenvalues are given by eq. (A.22), as is shown in the appendix. We transform from the $C^{(\pm)}$ basis to our basis and get

$$S_\kappa = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & \frac{1}{\cosh(\frac{\kappa\pi}{2})}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} = \frac{1}{2\cosh(\frac{\kappa\pi}{2})}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\,, \tag{2.27}$$

therefore, for the butterfly $u = \frac{1}{2\cosh(\frac{\kappa\pi}{2})}$, $v = 0$.

The nothing is the squeezed state defined by the identity matrix, $S = I$, therefore

$$S_\kappa = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\,, \tag{2.28}$$

that is $u = 1, v = -1$.

3 Relations to other representations

Now that we have the form of the $\mathcal{H}_{\kappa^2}$ projectors in the K_1 basis we would like to represent them in other forms. We start by transforming the states in $\mathcal{H}_{\kappa^2}$ to the oscillator basis. Then we derive a procedure to find the conformal map of squeezed states. Only surface state can pass this procedure, giving us a way to recognize which squeezed states are surface states. We demonstrate this procedure by some examples.

We also derive an inverse procedure for checking if a state is in $\mathcal{H}_{\kappa^2}$ and if so find the matrix S_κ representing it. Finding the spectrum given S_κ is straightforward. We use this procedure to show that the generalized butterfly states are in $\mathcal{H}_{\kappa^2}$ and to find their spectrum. Then we show that not all the L_{-2m} projectors are in $\mathcal{H}_{\kappa^2}$.

We also compare the structure of the states in $\mathcal{H}_{\kappa^2}$ to their form in the half-string formalism. Transforming S_κ to the continuous Moyal representation is straightforward and useful. We show that all the projectors in $\mathcal{H}_{\kappa^2}$ have the same normalization in this formalism and that the generalized butterfly states are orthogonal. We also transform S_κ to the discrete Moyal representation and show that projectors in $\mathcal{H}_{\kappa^2}$ satisfy the projector condition in this formalism.

3.1 The oscillator basis

To calculate the matrices in the oscillator basis of states in $\mathcal{H}_{\kappa^2}$ we use the orthogonality and completeness of the κ basis [**?**]. The basis vectors are defined by

$$|\kappa) = \sum_{n=1}^{\infty} v_n^{(\kappa)} \, |n) \ , \tag{3.1}$$

where

$$(n| = |n)^T = (0, 0, \dots, 0, 1, 0, 0, \dots) \, , \tag{3.2}$$

with the 1 in the n^{th} position of the vector. The coefficients

$$v_n^{(\kappa)} = (\kappa|n) \ , \tag{3.3}$$

are given by the generating function

$$\sum_{n=1}^{\infty} \frac{v_n^{(\kappa)}}{\sqrt{n}} z^n = \frac{1}{\kappa}(1 - e^{-\kappa \tan^{-1}(z)}) \equiv f_\kappa(z) \, . \tag{3.4}$$

The above relation can be inverted to give

$$v_n^{(\kappa)} = -\frac{\sqrt{n}}{2\pi i \kappa} \oint \frac{f_\kappa(z)}{z^{n+1}} dz = \frac{1}{\sqrt{n}} \frac{1}{2\pi i} \oint \frac{dz}{z^n} \partial_z f_\kappa(z) \, , \tag{3.5}$$

where the contour is around the origin, and we integrated by parts to get the last expression. The orthogonality and completeness relations in the κ basis are

$$(\kappa|\,\kappa') = \mathcal{N}(\kappa)\delta(\kappa-\kappa')\,, \tag{3.6}$$
$$\mathbf{1} = \int_{-\infty}^{\infty} d\kappa\, \frac{|\kappa)\,(\kappa|}{\mathcal{N}(\kappa)}\,, \tag{3.7}$$

where

$$\mathcal{N}(\kappa) = \frac{2}{\kappa}\sinh\left(\frac{\kappa\pi}{2}\right)\,, \tag{3.8}$$

is a normalization factor.

The matrix elements of a squeezed state can be written using a two-parameter generating function

$$S_{nm} = \frac{1}{\sqrt{nm}}\oint \frac{dzdw}{(2\pi i)^2}\frac{1}{z^n w^m}S(z,w)\,, \tag{3.9}$$

with the inverse relation

$$S(z,w) = \sum_{m,n=1}^{\infty}\sqrt{nm}S_{nm}z^{n-1}w^{m-1}\,. \tag{3.10}$$

The symmetry $S_{nm}=S_{mn}$ translates to the symmetry $S(z,w)=S(w,z)$.

We can now calculate the matrix elements of a state in the $\mathcal{H}_{\kappa^2}$ subalgebra. That is, given S_κ for $\kappa>0$ the matrix elements are

$$\begin{aligned}
S_{nm} = (n|\,S\,|m) = \int_{-\infty}^{\infty}\frac{d\kappa d\kappa'}{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}\,(n|\,\kappa)\,(\kappa|\,S\,|\kappa')\,(\kappa'|\,m) = \\
\int_0^{\infty}\frac{d\kappa d\kappa'}{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}\begin{pmatrix}(n|\,{-\kappa}) & (n|\,\kappa)\end{pmatrix}\begin{pmatrix}(-\kappa|\,S\,|{-\kappa'}) & (-\kappa|\,S\,|\kappa') \\ (\kappa|\,S\,|{-\kappa'}) & (\kappa|\,S\,|\kappa')\end{pmatrix}\begin{pmatrix}(-\kappa'|\,m) \\ (\kappa'|\,m)\end{pmatrix} = \\
\int_0^{\infty}\frac{d\kappa}{\mathcal{N}(\kappa)}\begin{pmatrix}(n|\,{-\kappa}) & (n|\,\kappa)\end{pmatrix}\begin{pmatrix}s_1(\kappa) & s_2(\kappa) \\ s_2(\kappa) & s_3(\kappa)\end{pmatrix}\begin{pmatrix}(-\kappa|\,m) \\ (\kappa|\,m)\end{pmatrix} = \\
\frac{1}{\sqrt{nm}}\oint\frac{dzdw}{(2\pi i)^2}\frac{1}{z^n w^m}\int_0^{\infty}\frac{d\kappa}{\mathcal{N}(\kappa)}\begin{pmatrix}\partial_z f_{-\kappa}(z) & \partial_z f_\kappa(z)\end{pmatrix}S_\kappa\begin{pmatrix}\partial_w f_{-\kappa}(w) \\ \partial_w f_\kappa(w)\end{pmatrix}\,.
\end{aligned} \tag{3.11}$$

Notice that we limited ourselves to the $\mathcal{H}_{\kappa^2}$ subalgebra in passing from the second line to the third line. In the last equality we used (3.5). By defining

$$s_{13}(\kappa) \equiv \begin{cases} s_3(\kappa) & \kappa > 0 \\ s_1(-\kappa) & \kappa < 0 \end{cases}, \qquad s_{22}(\kappa) \equiv \begin{cases} s_2(\kappa) & \kappa > 0 \\ s_2(-\kappa) & \kappa < 0 \end{cases}, \tag{3.12}$$

we get

$$\begin{aligned} S(z,w) = \int_{-\infty}^{\infty} \frac{d\kappa}{\mathcal{N}(\kappa)} \left(\partial_z f_\kappa(z) \partial_w f_\kappa(w) s_{13}(\kappa) + \partial_z f_\kappa(z) \partial_w f_{-\kappa}(w) s_{22}(\kappa) \right) = \\ \int_{-\infty}^{\infty} \frac{d\kappa}{\mathcal{N}(\kappa)} \frac{e^{-\kappa(\tan^{-1}(z)+\tan^{-1}(w))} s_{13}(\kappa) + e^{\kappa(\tan^{-1}(w)-\tan^{-1}(z))} s_{22}(\kappa)}{(1+z^2)(1+w^2)} \,. \end{aligned} \tag{3.13}$$

This equation is what we were after: the oscillator matrix elements as a function of S_κ.

3.2 Surface states

The matrix elements expression (3.9) for squeezed states has a similar structure to that of surface states [**?**, **?**, **?**]

$$S_{nm} = -\frac{1}{\sqrt{nm}} \oint \frac{dzdw}{(2\pi i)^2 z^n w^m} \frac{f'(-z) f'(-w)}{(f(-z) - f(-w))^2} \,, \tag{3.14}$$

where $f(z)$ is the conformal transformation that defines the surface state. This raises the question which squeezed states are surface states, and what is their conformal transformation $f(z)$. Squeezed states are surface states if and only if $\exists f(z)$ such that

$$S(z,w) \approx -\frac{f'(-z) f'(-w)}{(f(-z) - f(-w))^2} \,. \tag{3.15}$$

By $\approx$ we mean "equal up to poles", since poles do not contribute to the contour integrals in eq. (3.14). However $S(z,w)$ is regular near the origin, and

$$\frac{f'(-z) f'(-w)}{(f(-z) - f(-w))^2} = \frac{1}{(z-w)^2} + \text{regular terms} \,. \tag{3.16}$$

Therefore, the condition is that there exists $f(z)$ such that

$$S(z,w) = -\frac{f'(-z)f'(-w)}{(f(-z)-f(-w))^2} + \frac{1}{(z-w)^2}\,. \tag{3.17}$$

If this is the case, then in particular

$$S(z,0) = -\frac{f'(-z)}{f(-z)^2} + \frac{1}{z^2}\,, \tag{3.18}$$

where we used $f(0)=0, f'(0)=1$ which is possible due to $SL(2,\mathbb{C})$ invariance. The solution of this equation gives us a candidate for $f(z)$

$$f^{c}(z) = \frac{z}{1 - z\int_0^{-z} S(\tilde{z},0)d\tilde{z}}. \tag{3.19}$$

Given $S(z,w)$, one can solve eq. (3.19) to get $f^{c}(z)$, then substitute the solution in eq. (3.17) and check if it reproduces $S(z,w)$. A squeezed state is a surface state if and only if $f^{c}(z)$ reproduces $S(z,w)$. We now turn to some examples.

3.2.1 Reconstructing the butterfly

To evaluate the generating function (3.13) of the butterfly (2.27) we use the relation

$$\int_{-\infty}^{\infty} d\kappa \frac{e^{c\kappa}}{\frac{2}{\kappa}\sinh\left(\frac{\kappa\pi}{2}\right) 2\cosh(\frac{\kappa\pi}{2})} = \frac{1}{4\cos(\frac{c}{2})^2}\,. \tag{3.20}$$

Replacing c by $-\tan^{-1}(z) \pm \tan^{-1}(w)$ we obtain

$$S(z,w) = \frac{w^2+z^2-\frac{w^2+z^2+2w^2z^2}{\sqrt{(1+w^2)(1+z^2)}}}{(w^2-z^2)^2}\,. \tag{3.21}$$

By eq. (3.19), we get

$$f(z) = \frac{z}{\sqrt{1+z^2}}\,, \tag{3.22}$$

which is the correct expression for the butterfly. Substituting this expression into (3.17) reproduces eq. (3.21), as it should.

3.2.2 The duals of the nothing and of the identity states

We demonstrated in 2.3 that the nothing state has $u = 1, v = -1$, that is $S_\kappa = \mathbf{1}$. The dual of the nothing has $S_\kappa = -\mathbf{1}$, and is the mirror image of the nothing in the u, v plane, $u = v = -1$. Both projectors live at the boundary of figure 1, and saturate the inequality (3.17) for all κ, and are therefore very singular. It can be shown that the mirror images of the other butterfly states do not correspond to continuous projectors.

Equation (3.13) now gives

$$S(z, w) = -\frac{1}{(1 - wz)^2} \,. \tag{3.23}$$

Now we use eq. (3.19) to find the candidate $f(z)$

$$f^{\rm c}(z) = \frac{z}{1 - z^2} \,. \tag{3.24}$$

We recognize $f^{\rm c}(z)$ as the conformal transformation of the identity state, which is represented by C, rather by $-I$. Indeed, when we substitute (3.24) back into (3.17) we get

$$S^{\rm Id}(z, w) = -\frac{1}{(1 + wz)^2} \,, \tag{3.25}$$

instead of (3.23). This proves that the dual of the nothing state is not a surface state.

The dual of the identity $S_\kappa = -C_\kappa$ is not a projector. It has

$$f^{\rm c}(z) = \frac{z}{1 + z^2} \,, \tag{3.26}$$

which is the conformal map of the nothing, meaning that it is also not a surface state.

3.2.3 A non-orthogonal projector

Here we want to give an example of another surface state projector. This projector has $v = 0$, $u = \frac{1}{2}$ for all κ. Notice that it does not respect the reality condition (2.14) meaning that it is a non-orthogonal projector as is discussed in section (3.4). In the K_1 basis the projector is

$$s_{13}(\kappa) = \frac{1}{2}\left(1 \pm \tanh\left(\frac{\kappa\pi}{2}\right)\right) , \tag{3.27}$$

$$s_{22}(\kappa) = \frac{1}{2\cosh\left(\frac{\kappa\pi}{2}\right)} . \tag{3.28}$$

Note that we are describing two projectors ($\pm$). Non-orthogonal projectors come in pairs, because they are asymmetric. Therefore, if we find one projector, its twisted partner will give another. The conformal transformations of the projectors are

$$f(z) = \pm 1 + \frac{z \mp 1}{\sqrt{1+z^2}} . \tag{3.29}$$

We can alway build two orthogonal projectors from a non-orthogonal pair by star multiplying them in different orders. The current states are a gluing of the generalized butterfly ($\alpha = \frac{2}{3}$) on one side with the nothing ($\alpha = 2$) on the other [?].

3.3 The inverse transformation

In the previous subsections we constructed the matrix representation and the conformal map representation for states in the $\mathcal{H}_{\kappa^2}$ subalgebra. It is natural to ask the opposite question. Does a given state belongs to $\mathcal{H}_{\kappa^2}$, and if so, what is its form in the K_1 basis. Given its form in the K_1 basis we can immediately check if it is a projector (2.16).

In the appendix we do it explicitly for the butterfly. Here we will find a general prescription for all states by examining when is it possible to invert the relations of the previous subsection.

Given $f(z)$ or S_{nm} we can use eq. (3.17), or eq. (3.10) to define $S(z,w)$. The question is, what are the conditions for the existence of $s_{13}(\kappa)$, $s_{22}(\kappa)$ which reproduce $S(z,w)$ via eq. (3.13), and what are those $s_{13}(\kappa)$, $s_{22}(\kappa)$.

Inspecting eq. (3.13), we see that $S(z,w)(1+z^2)(1+w^2)$ cannot be an arbitrary function of z, w for states in the $\mathcal{H}_{\kappa^2}$ subalgebra, but should rather be a sum of two terms

$$S(z,w)(1+z^2)(1+w^2) = F_1(\xi) + F_2(\zeta) , \tag{3.30}$$

where we have defined

$$\begin{aligned} \xi &= i(\tan^{-1}(z) + \tan^{-1}(w))\,, \\ \zeta &= i(\tan^{-1}(z) - \tan^{-1}(w))\,. \end{aligned} \tag{3.31}$$

From eq. (3.13) we see that $F_1(\xi)$ is the Fourier transform of $s_{13}(\kappa)/\mathcal{N}(\kappa)$, while $F_2(\zeta)$ is the Fourier transform of $s_{22}(\kappa)/\mathcal{N}(\kappa)$. We conclude that this split to a sum of functions is a necessary condition for the state to be in $\mathcal{H}_{\kappa^2}$ due to the form of eq. (3.13), and a sufficient condition since the Fourier transform is invertible.

Suppose now that F_1 and F_2 are given. The inverse Fourier Transform reads

$$\begin{aligned} s_{13}(\kappa) &= \frac{\mathcal{N}(\kappa)}{2\pi} \int_{-\infty}^{\infty} e^{-i\kappa\xi} F_1(\xi)\, d\xi\,, \\ s_{22}(\kappa) &= \frac{\mathcal{N}(\kappa)}{2\pi} \int_{-\infty}^{\infty} e^{-i\kappa\zeta} F_2(\zeta)\, d\zeta\,. \end{aligned} \tag{3.32}$$

These are the desired expressions for inverting the transformation. They apply only to states in $\mathcal{H}_{\kappa^2}$, meaning, states of the form (3.30). We can use eq. (3.32,2.16) to check if a given state is a projector, as is illustrated below.

3.3.1 The spectrum of the generalized butterfly states

The generalized butterfly states are a one-parameter family of projectors [**?**]. They are defined by the maps

$$f(z) = \frac{1}{\alpha} \sin(\alpha \tan^{-1}(z))\,, \tag{3.33}$$

where $0 \leq \alpha \leq 2$. Special cases include the sliver $\alpha = 0$, for which

$$f(z) = \tan^{-1}(z)\,, \tag{3.34}$$

the canonical butterfly $\alpha = 1$ (3.22), and the nothing state $\alpha = 2$ with

$$f(z) = \frac{z}{1+z^2}\,. \tag{3.35}$$

We substitute the map (3.33) in eq.(3.17), and notice that indeed it splits according to eq. (3.30), with

$$
\begin{aligned}
F_1(\xi) &= \frac{\alpha^2}{4\cosh\left(\frac{\alpha\xi}{2}\right)^2}\,, \\
F_2(\zeta) &= \frac{\alpha^2}{4\sinh\left(\frac{\alpha\zeta}{2}\right)^2} - \frac{1}{\sinh(\zeta)^2}\,,
\end{aligned}
\tag{3.36}
$$

which means that the generalized butterfly states are in $\mathcal{H}_{\kappa^2}$. Using eq. (3.32) we get

$$
\begin{aligned}
s_1 = s_3 = s_{13} &= \frac{\sinh\left(\frac{\kappa\pi}{2}\right)}{\sinh\left(\frac{\kappa\pi}{\alpha}\right)}\,, \\
s_2 = s_{22} &= \cosh\left(\frac{\kappa\pi}{2}\right) - \coth\left(\frac{\kappa\pi}{\alpha}\right)\sinh\left(\frac{\kappa\pi}{2}\right)\,.
\end{aligned}
\tag{3.37}
$$

To evaluate the integrals, we closed the integration contour with a semi-circle around the lower half plane, picking up an infinite number of residues along the lower half of the imaginary axis, which summed up to the given results.

A simple consistency check shows that these matrix elements satisfy eq. (2.16), which verifies that the generalized butterfly states are indeed projectors.

3.3.2 The L_{-2m} projectors

Another class of projectors with a simple Virasoro representation was introduced in [?]. These projectors are surface states defined by the conformal maps

$$
f_m(z) = \frac{z}{(1-(-z^2)^m)^{1/2m}}\,,
\tag{3.38}
$$

with associated states

$$
|P_{2m}\rangle = \exp\left(\frac{(-1)^m}{2m}L_{-2m}\right)|0\rangle\ .
\tag{3.39}
$$

In order to check if a given surface state is in $\mathcal{H}_{\kappa^2}$, we found it useful to use a condition which is equivalent to eq. (3.30) for surface states

$$
\Diamond\Box\log\left(\frac{f(-z)-f(-w)}{z-w}\right) = 0\,,
\tag{3.40}
$$

where $\Box = \frac{d^2}{d\xi^2} - \frac{d^2}{d\zeta^2}$ is the two dimensional d'Alembertian and $\Diamond = \frac{d^2}{d\xi\, d\zeta}$ is the light-cone d'Alembertian, or vice versa. The relation between z, w and ξ, ζ is given by eq. (3.31). This condition holds, as expected, for the case of the butterfly $m = 1$. A direct check shows that this equation does not hold for the next 100 cases and therefore that these projectors are not in the $\mathcal{H}_{\kappa^2}$ subalgebra. We expect that higher m projectors will not fulfill this condition either.

3.4 Half-string representation

In this subsection we discuss how states in the $\mathcal{H}_{\kappa^2}$ subalgebra look in the half-string formalism [?, ?, ?, ?]. Finding rank one projectors in half-string formalism is extremely simple. Every normalized string wave functional $\Psi_P[x(\sigma)]$ that has a factorized form

$$\Psi_P[x(\sigma)] = \chi_1[l(\sigma)]\chi_2[r(\sigma)]\,, \tag{3.41}$$

where $l(\sigma) \sim x(\sigma)$ and $r(\sigma) \sim x(\pi - \sigma)$ are the left and right sides of the string [1], corresponds to a rank one projector. The string field reality condition (2.12) for a string functional reads

$$\Psi_P^*[x(\sigma)] = \Psi_P[x(\pi - \sigma)] \Rightarrow \chi_2[r(\sigma)] = \chi_1^*[r(\sigma)]\,, \tag{3.42}$$

leaving us with orthogonal projectors only. In [?, ?] projectors with real Gaussian functionals were considered. We consider also complex Gaussians because they appear in $\mathcal{H}_{\kappa^2}$

$$\begin{aligned}\Psi[X(\sigma)] &= \exp\left(-\frac{1}{2} l_{2k-1} M_{2k-1,2j-1} l_{2j-1}\right) \exp\left(-\frac{1}{2} r_{2k-1} M^*_{2k-1,2j-1} r_{2j-1}\right)\\ &= \exp\left(-\frac{1}{2} x_n L_{nm} x_m\right)\,,\end{aligned} \tag{3.43}$$

where l_{2k+1}, r_{2j+1} and x_n are the Fourier modes of $l(\sigma), r(\sigma)$ and $x(\sigma)$. The form of the Gaussians in the half-string formalism was restricted by the string field reality condition. For the full-string matrix L this condition reads

$$L^* = CLC\,. \tag{3.44}$$

[1] There are different approaches in defining half string coordinates according to whether or not the midpoint $x(\pi/2)$ is subtracted [?, ?]. This determines the half string boundary conditions and is related to the associativity anomaly [?] of the star product.

Solving eq. (3.43) for L gives

$$\begin{aligned} L_{2n-1,2m-1} &= 2\,\mathrm{Re}\, M_{2n-1,2m-1}\,, \\ L_{2n,2m} &= 2\,(T\,\mathrm{Re}\, M\, T^T)_{2n,2m}\,, \\ L_{2n,2m-1} = L_{2m-1,2n} &= 2i\,(T\,\mathrm{Im}\, M)_{2n,2m-1}\,, \end{aligned} \tag{3.45}$$

where the matrix T is defined by

$$\begin{aligned} T_{2n,2m-1} =& \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos(2n\sigma)\cos((2m-1)\,\sigma)\,d\sigma = \\ =& \frac{2(-1)^{m+n+1}}{\pi}\left(\frac{1}{2m-1+2n}+\frac{1}{2m-1-2n}\right)\,. \end{aligned} \tag{3.46}$$

The relation between Gaussian functionals and squeezed states is given by

$$S = \frac{1-ELE}{1+ELE}\,, \tag{3.47}$$

where the diagonal matrix E is defined as in [**?**].

The case of a twist invariant projector is given by a real matrix M, which corresponds to a real, block diagonal S_{nm}. The form of S_{nm} restricts the Taylor expansion of $S(z,w)$ to monomials $z^k w^l$ with real coefficients, where k and l are both odd or both even. Inspection of the symmetry properties of the integrand in eq. (3.13) shows that this requirement is satisfied only if $s_3(\kappa) = s_1(\kappa)$ which is the twist invariance condition (2.20).

The case of a general orthogonal projector, i.e. a BPZ real one, eq. (3.42), is given by a complex matrix M. In this case S_{nm} can have imaginary entries in the odd–even blocks. The analog condition for states in $\mathcal{H}_{\kappa^2}$ is $s_3(\kappa) = s_1^*(\kappa)$. Non-orthogonal, non-real projectors can have general s_1, s_2, s_3 and general S_{nm} matrices limited only by the projection condition.

3.5 Moyal representation

The star algebra can be represented as an infinite tensor product of two dimensional Moyal spaces. Two such representation were found. The discrete one by Bars [**?**] (see also Bars and Matsuo [**?**]), and the continuous one by Douglas, Liu, Moore and Zwiebach [**?**] (DLMZ below). Both are intimately related to the K_1 basis.

We shall start with the continuous Moyal representation and then turn to the discrete one.

3.5.1 Continuous Moyal representation

In this representation for each $\kappa > 0$ there is a pair of noncommutative coordinates x_κ, y_κ with $*$ commutation relation

$$[x_\kappa, y_{\kappa'}]_* = \theta(\kappa)\delta(\kappa - \kappa')\,, \tag{3.48}$$

where

$$\theta(\kappa) = 2\tanh\left(\frac{\kappa\pi}{4}\right)\,, \tag{3.49}$$

is the noncommutativity parameter. These two coordinates are related to creation and annihilation operators by

$$x_\kappa = \frac{i}{\sqrt{2}}(e_\kappa - e_\kappa^\dagger)\,, \qquad y_\kappa = \frac{i}{\sqrt{2}}(o_\kappa - o_\kappa^\dagger)\,, \tag{3.50}$$

and $e_\kappa^\dagger, (o_\kappa^\dagger)$ are related to the twist even (odd) creation operators by

$$e_\kappa^\dagger = \sqrt{2}\sum_{n=1}^{\infty}\frac{v_{2n}^{(\kappa)}}{\sqrt{\mathcal{N}}}a_{2n}^\dagger\,, \qquad o_\kappa^\dagger = -i\sqrt{2}\sum_{n=1}^{\infty}\frac{v_{2n-1}^{(\kappa)}}{\sqrt{\mathcal{N}}}a_{2n-1}^\dagger\,, \tag{3.51}$$

where $v_n^{(\kappa)}, \mathcal{N}$ are given by eq. (3.5,3.8). Note that $v_n(\kappa)$ in DLMZ is equal to $\frac{v_n^{(\kappa)}}{\sqrt{\mathcal{N}}}$ here.

From this definition it is straightforward to transform the matrix S_κ of the $\mathcal{H}_{\kappa^2}$ subalgebra to the $e_\kappa^\dagger, o_\kappa^\dagger$ basis

$$\tilde{S}_\kappa = \begin{pmatrix} s_{ee} & is_{eo} \\ is_{eo} & -s_{oo} \end{pmatrix}\,, \tag{3.52}$$

where

$$\begin{pmatrix} s_{ee} & s_{eo} \\ s_{eo} & s_{oo} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\,. \tag{3.53}$$

Note the i factor in the definition of $o_\kappa^\dagger$ in eq. (3.51), which was put there to make the BPZ conjugation relations of o_κ and e_κ the same

$$BPZ(e_\kappa) = -e_\kappa^\dagger\,, \qquad BPZ(o_\kappa) = -o_\kappa^\dagger\,. \tag{3.54}$$

This is the source of the i factor in eq. (3.52), which when combined with the reality condition (2.14), translates to the reality of the matrix $\tilde{S}_\kappa$. The projection condition (2.16) now takes the form

$$1+\det(\tilde{S}_\kappa)+\cosh\left(\frac{\kappa\pi}{2}\right)\mathrm{tr}(\tilde{S}_\kappa)=0\,. \tag{3.55}$$

The matrix $\tilde{S}_\kappa$ corresponds in coordinate space to the matrix

$$\tilde{L}_\kappa=\frac{1-\tilde{S}_\kappa}{1+\tilde{S}_\kappa}\,, \tag{3.56}$$

where the wave function in $\vec{X}_\kappa\equiv(x_\kappa,y_\kappa)$ is proportional to the Gaussian

$$\exp\left(-\frac{1}{2}\int_0^\infty \vec{X}_\kappa\tilde{L}_\kappa\vec{X}_\kappa d\kappa\right)\,. \tag{3.57}$$

The generalized Butterfly (3.37) in this representation is

$$\tilde{L}_\kappa=\coth\left(\frac{\kappa\pi}{4}\right)\begin{pmatrix}\tanh(\frac{\kappa\pi(2-\alpha)}{4\alpha}) & 0\\ 0 & \coth(\frac{\kappa\pi(2-\alpha)}{4\alpha})\end{pmatrix}\,. \tag{3.58}$$

We can think of these solutions as noncommutative solitons [**?**, **?**]. In the sliver limit $\alpha\to 0$ we get

$$\tilde{L}_\kappa=\coth\left(\frac{\kappa\pi}{4}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\,, \tag{3.59}$$

in agreement with [**?**]. The generalized Butterfly states can now be identified as non-radial generalizations of the sliver, with an opposite rescaling of x_κ, y_κ, which is α and κ dependent. Rescaling a projector gives again a projector because it does not change the brackets (3.48). All our twist invariant solutions (2.20) are of this type.

Plugging the inverse of eq. (3.56) into the projector condition (3.55), one gets the elegant condition

$$\det(\tilde{L}_\kappa)=\coth^2\left(\frac{\kappa\pi}{4}\right)=\frac{4}{\theta^2(\kappa)}\,. \tag{3.60}$$

We now show that the projectors of $\mathcal{H}_{\kappa^2}$ can be interpreted as a change of coordinates which does not change the brackets (3.48). Under a change of coordinates $\vec{X} \to U\vec{X}$, we get $\tilde{L}_\kappa \to U^{-1^T}\tilde{L}_\kappa U^{-1}$. The reality of the change of coordinates, together with the invariance of the brackets means that $U \in SP(2,\mathbb{R})$. The reality and positivity of $\tilde{L}_\kappa$ together with the invariance of $\det(\tilde{L}_\kappa)$ means that $U \in SL(2,\mathbb{R})$. But these two groups are the same. Note that $SP(2,\mathbb{R})$ has three generators, while $\tilde{L}_\kappa$ has only two degrees of freedom. For each $\tilde{L}_\kappa$ there exists a generator of $SP(2,\mathbb{R})$, which acts trivially on it.

$\tilde{L}_\kappa$ can be parametrized as

$$\tilde{L}_\kappa = \begin{pmatrix} l_1^{(\kappa)} & l_2^{(\kappa)} \\ l_2^{(\kappa)} & l_3^{(\kappa)} \end{pmatrix} = \coth\left(\frac{\kappa\pi}{4}\right) \begin{pmatrix} r(\kappa)^{-1}\cosh(\phi(\kappa)) & \sinh(\phi(\kappa)) \\ \sinh(\phi(\kappa)) & r(\kappa)\cosh(\phi(\kappa)) \end{pmatrix} . \tag{3.61}$$

A continuous solution corresponds to a continuous choice of $r(\kappa), \phi(\kappa)$. Continuity at $\kappa = 0$ translates into the requirement that as κ goes to zero the term $\coth(\frac{\kappa\pi}{4})r(\kappa)\cosh(\phi(\kappa))$ either diverges ($s_0 = 1$), or goes to zero ($s_0 = -1$).

This can be generalized to a change of coordinates, for which $\mathcal{H}_{\kappa^2}$ is not invariant, that mixes x_κ, y_κ for different values of κ without changing the brackets. Some kind of continuity should be imposed here as well, in order to define this group of transformations. With the notion of continuity made clear, this more general symmetry can be used for the production of general Gaussian projectors. Presumably all rank one Gaussian projectors (such as the L_{-2n}, which are not in $\mathcal{H}_{\kappa^2}$) can be represented this way. We shall return to this point in section 3.5.2. When non-Gaussian projectors are included, they are connected by yet another gauge group [**?**]. The most general gauge symmetry, which connects different rank-one projectors, should be generated by these two groups.

To normalize the $\mathcal{H}_{\kappa^2}$ projectors in the DLMZ conventions we calculate

$$\int Dx(\kappa)Dy(\kappa)\exp\left(-\int_0^\infty d\kappa \vec{X}_\kappa \tilde{L}_\kappa \vec{X}_\kappa\right) = \\ \exp\left(-\frac{1}{4}\frac{\log L}{2\pi}\int_0^\infty d\kappa \log\det(\tilde{L}_\kappa)\right) = \exp\left(-\frac{\log L}{8}\right) , \tag{3.62}$$

where L is the level cutoff, using eq. (3.60). The normalized state is then given by

$$\exp\left(\frac{\log L}{16}\right)\exp\left(-\frac{1}{2}\int_0^\infty \vec{X}_\kappa \tilde{L}_\kappa \vec{X}_\kappa d\kappa\right) . \tag{3.63}$$

The normalization factor is infinite, since we should take the cutoff $L \to \infty$. It is usually claimed that this would be compensated by the ghost factor, although it may be not that simple [**?**, **?**].

Using this normalization we now show that, at least in the matter sector, different Gaussian projectors are orthogonal, and thus can be used to construct multi D-brane solutions with no need to include non-Gaussian projectors [**?**, **?**]. Again the ghost sector may play a role, since the inclusion of "unbalanced states" may be problematic [**?**]. The inner product of two arbitrary generalized Butterfly states parametrized by $a_i \equiv \frac{2-\alpha_i}{\alpha_i}$ is

$$\exp\left(\frac{\log L}{8}\right)\int Dx(\kappa)Dy(\kappa)\exp\left(-\int_0^\infty d\kappa\frac{1}{2}\vec{X}_\kappa(\tilde{L}_\kappa^{a_1}+\tilde{L}_\kappa^{a_2})\vec{X}\kappa\right) =$$
$$\exp\left(\frac{\log L}{8}-\frac{1}{4}\frac{\log L}{2\pi}\int_0^\infty d\kappa\log\det\frac{\tilde{L}_\kappa^{a_1}+\tilde{L}_\kappa^{a_2}}{2}\right) =$$
$$\exp\left(-\frac{\log L}{8\pi}\int_0^\infty d\kappa\log\frac{2+\tanh(a_1\kappa)\coth(a_2\kappa)+\tanh(a_2\kappa)\coth(a_1\kappa)}{4}\right) =$$
$$\exp\left(-\frac{\log L}{48}\frac{(a-b)^2}{ab(a+b)}\right)\,. \qquad (3.64)$$

For $a \neq b$ the inner product goes to zero when the cutoff L is sent to infinity. This orthogonality of two Gaussians is only possible due to the fact that our space of integration is infinite dimensional. Indeed for any finite L the overlap is nonzero, and thus the orthogonality is only approximate at a cut-off theory.

3.5.2 Discrete Moyal representation

The basic $*$ commutation relation in this representation [**?**, **?**] is

$$[x_{2n}, p_{2m}]_* = i\theta\delta_{m,n}\,, \qquad (3.65)$$

where θ is an arbitrary parameter, x_{2n} are the even modes of the string, and p_{2n} are linear combinations of the odd momenta of the string

$$p_{2m} = \frac{\theta}{2}\sum_{n=1}^{\infty}p_{2n-1}R_{2n-1,2m}\,, \qquad p_{2n-1} = \frac{2}{\theta}\sum_{n=1}^{\infty}p_{2m}T_{2m,2n-1}\,. \qquad (3.66)$$

The matrix R is the inverse of the matrix T (3.46).

$\mathcal{H}_{\kappa^2}$ states should be Gaussians in the variables x_{2n}, p_{2n}. To see their form in this formulation we can use the transformation to the oscillator basis eq. (3.11), and then use the inverse of eq. (3.47), Fourier transform the odd modes, and use the T matrix to get the final form. The other option would be to use the form of $\mathcal{H}_{\kappa^2}$ states in the continuous Moyal representation, and then transform them to the discrete one [**?**]. We shall use this method in order to avoid the technical problems of inverting infinite dimensional matrices. We shall need eq. (4.11,4.12) of DLMZ, which we present here in our notations

$$\begin{aligned} x_\kappa = & \sqrt{2}\sum_{n=1}^{\infty} \frac{v_{2n}^{(\kappa)}\sqrt{2n}}{\sqrt{\mathcal{N}(\kappa)}} x_{2n}\,, \\ y_\kappa = - & \sqrt{2}\sum_{n=1}^{\infty} \frac{v_{2n-1}^{(\kappa)}}{\sqrt{\mathcal{N}(\kappa)}\sqrt{2n-1}} p_{2n-1}\,. \end{aligned} \tag{3.67}$$

Squeezed states in the discrete Moyal representation read

$$\exp(-\xi_i M_{ij} \xi_j)\,, \tag{3.68}$$

where M is an infinite matrix and $\xi = (x_2, x_4, ..., p_2, p_4, ...)$. Our task is to find the transformation from $\tilde{L}_\kappa$ to M using

$$\exp(-\xi M \xi) = \exp\left(-\frac{1}{2}\int_0^\infty \vec{X}_\kappa \tilde{L}_\kappa \vec{X}_\kappa d\kappa\right)\,. \tag{3.69}$$

We write the matrix M as

$$M = \begin{pmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{pmatrix}\,, \tag{3.70}$$

where M_1, M_2, M_3 are infinite matrices, and M_1, M_3 are symmetric. Now we substitute eq. (3.67,3.66) into eq. (3.69), and compare coefficients to get

$$
\begin{aligned}
(M_1)_{2m,2n} &= \int_0^\infty d\kappa \frac{v_{2m}^{(\kappa)} v_{2n}^{(\kappa)} \sqrt{(2m)(2n)}}{\mathcal{N}(\kappa)} l_1^{(\kappa)} \,, \\
(M_2)_{2m,2n} &= -\int_0^\infty d\kappa \frac{v_{2m}^{(\kappa)} \sqrt{2m}}{\mathcal{N}(\kappa)} l_2^{(\kappa)} \frac{2}{\theta} \sum_{k=1}^{\infty} T_{2n,2k-1} \frac{v_{2k-1}^{(\kappa)}}{\sqrt{2k-1}} = \\
&= -\frac{2}{\theta} \int_0^\infty d\kappa \frac{v_{2m}^{(\kappa)} v_{2n}^{(\kappa)} \sqrt{2m}}{\mathcal{N}(\kappa)\sqrt{2n}} l_2^{(\kappa)} \tanh\left(\frac{\kappa\pi}{4}\right) \,, \\
(M_3)_{2m,2n} &= \int_0^\infty d\kappa \frac{l_3^{(\kappa)}}{\mathcal{N}(\kappa)} \frac{4}{\theta^2} \sum_{k,l=1}^{\infty} T_{2m,2l-1} T_{2n,2k-1} \frac{v_{2l-1}^{(\kappa)} v_{2k-1}^{(\kappa)}}{\sqrt{(2l-1)(2k-1)}} = \\
&= \frac{4}{\theta^2} \int_0^\infty d\kappa \frac{v_{2m}^{(\kappa)} v_{2n}^{(\kappa)}}{\mathcal{N}(\kappa)\sqrt{2m}\sqrt{2n}} l_3^{(\kappa)} \tanh^2\left(\frac{\kappa\pi}{4}\right) \,.
\end{aligned}
\tag{3.71}
$$

In the case of M_2, M_3 we also used the identity

$$
\sum_{k=1}^{\infty} T_{2n,2k-1} \frac{v_{2k-1}^{(\kappa)}}{\sqrt{2k-1}} = -\frac{v_{2n}^{(\kappa)}}{\sqrt{2n}} \tanh\left(\frac{\kappa\pi}{4}\right) \,, \tag{3.72}
$$

which is an immediate consequence of eq. (6.7,6.8,3.10) of [**?**].

In [**?**] Bars and Matsuo have considered the subalgebra of (shifted) Gaussian states. This subalgebra has a form of a monoid, when the Gaussian matrices are not restricted to be positive, and the star-multiplication is formally defined in the same way for all the matrices. They have found that a state M in the monoid is a projector if either $M = 0$, in which case it represents the identity, or

$$
(M\sigma)^2 = \mathbf{1} \,, \tag{3.73}
$$

where σ is the noncommutativity matrix, given by

$$
\sigma = i\theta \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \,. \tag{3.74}
$$

The condition $(M\sigma)^2 = \mathbf{1}$ implies that θM is an element of $SP(\infty)$. This reminds us of the projector criteria we discussed at section 3.5.1. Of course, one has to well define the meaning of $SP(\infty)$, according to the class of matrices which are considered legitimate.

In terms of the matrices M_1, M_2, M_3, the condition (3.73) tells that the matrices M_2M_1 and M_3M_2 are symmetric, and that

$$M_1M_3 - M_2^2 = \mathbf{1}\,. \tag{3.75}$$

For states in $\mathcal{H}_{\kappa^2}$ the matrices M_2M_1 and M_3M_2 are always symmetric, regardless of the projector condition. As for the last condition, it is an immediate result of the form of M_1, M_2, M_3, eq. (3.71), of the orthogonality condition (3.6), and of the projector condition, eq. (3.60). We see that our results are indeed compatible with the general constraint, eq. (3.73).

4 Conclusions

In this paper we set out to find squeezed state projectors whose matrix commute with $(\mathrm{K}_1)^2$. The set $\mathcal{H}_{\kappa^2}$ of squeezed states that commute with $(\mathrm{K}_1)^2$ is a subalgebra of the star algebra. Analyzing this subalgebra is straightforward, using the explicit form of V_3 (2.10). This subalgebra obviously contains the wedge states, $\mathcal{H}_{wegde} \subset \mathcal{H}_{\kappa^2}$. The generalized butterfly states are also in $\mathcal{H}_{\kappa^2}$, but some other surface states are not. There are also states in $\mathcal{H}_{\kappa^2}$ which are not surface states.

All this makes $\mathcal{H}_{\kappa^2}$ a convenient laboratory for the study of the star algebra. The BPZ reality condition is given by eq. (2.14). The condition for twist invariance is then that the matrix is real, eq. (2.20). The projection condition also has a simple form, eq. (2.16).

The $\mathcal{H}_{\kappa^2}$ subalgebra has a very simple representation in the continuous Moyal formalism (3.52). Using this formalism we found that all the projectors in $\mathcal{H}_{\kappa^2}$ have the same (divergent) normalization (3.63). We used this normalization to show that any two generalized butterfly states are orthogonal (3.64).

It would be interesting to try to address some of the open problems of string field theory in the $\mathcal{H}_{\kappa^2}$ subalgebra.

- If all rank one projectors represent the same D-brane they should be related by a gauge transformation. The fact that all the projectors in $\mathcal{H}_{\kappa^2}$ have the same normalization and that we know what is the symmetry that relates them to each other, could be a step in the right direction.

- The allowed space of states in SFT is not well understood. In the $\mathcal{H}_{\kappa^2}$ subalgebra this question should translate to conditions on the functions of κ. Those functions must be integrable to have any meaning. But we do not know what other constraints they have and specifically we do not know if we should impose continuity.

Addressing the above questions would require an analysis of the ghost sector and of the zero-modes. These two issues were neglected in this paper. It might be possible to incorporate these issues in the $\mathcal{H}_{\kappa^2}$ subalgebra using the known spectrum of the full vertex.

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A Butterfly spectroscopy

In this appendix we show, independently of the methods developed in the main text, that the butterfly state [?] shares eigenvectors with the wedge states and compute its eigenvalues.

In the oscillator representation the butterfly is given as a squeezed state with the following Neumann coefficients

$$V^B_{mn} = \begin{cases} -(-1)^{\frac{m+n}{2}} \frac{\sqrt{mn}}{m+n} \frac{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}{\pi\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})} & m,n \text{ odd} \\ 0 & \text{otherwise} \end{cases} . \tag{A.1}$$

In [?] the spectrum of the wedge states T_n was found based on the commutation relation

$$[T_n, \mathrm{K}_1] = 0\,, \tag{A.2}$$

and the non-degeneracy of K_1's spectrum. Yet,

$$[V^B, K_1] \neq 0\,, \tag{A.3}$$

Thus our claim that the butterfly shares eigenvectors with the family of wedge states requires an explanation. The way out is to note that although K_1 is non-degenerate T_n are doubly degenerate. Therefore, there are two sets of commuting operators that can be diagonalized simultaneously. The first set includes K_1 and T_n while the second includes $(K_1)^2, T_n$ and C. We need to show that C and T_n commute with V^B in order to prove our claim, since C lifts the degeneracy of the eigenvectors of T_n. From the explicit form of V^B as given in (A.1) we see that V^B is twist invariant

$$[V^B, C] = 0\,, \tag{A.4}$$

and that all its twist even eigenvalues vanish

$$V^B v_+^{(\kappa)} = 0\,, \tag{A.5}$$

The squeezed states T_n can be represented as functions of a single matrix B defined by

$$T_{2+\epsilon} = \epsilon B + O(\epsilon^2)\,, \tag{A.6}$$

which was introduced in [?]. Its entries are

$$B_{mn} = \begin{cases} -\frac{(-1)^{\frac{n-m}{2}}\sqrt{mn}}{(m+n)^2-1} & m+n \text{ even} \\ 0 & \text{otherwise} \end{cases}\,. \tag{A.7}$$

The twist odd eigenvalues of B are non-degenerate.

In section A.1 we will rely on the simple form of the matrix B to prove that $[V^B, T_n] = 0$. We shall use this result in section A.2 in order to find the eigenvalues of the butterfly

A.1 The butterfly commutes with the B matrix

Both B and V^B are symmetric matrices. Two such matrices commute iff their product $G \equiv BV^B$ is a symmetric matrix. It is immediate that $G_{mn} = 0$

unless both m, n are odd. Now we use eq. (A.7,A.1) to find

$$G_{mn} = \sum_{l=1}^{\infty} B_{ml} V^B_{ln} =$$

$$\frac{2\sqrt{mn}(-1)^{\frac{n-m}{2}}\Gamma(\frac{n}{2})}{\pi\Gamma(\frac{n+1}{2})} \sum_{k=1}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k)(2k+n-1)((m+2k-1)^2-1)}\,, \quad \text{(A.8)}$$

where in the second line we used the fact that $V^B_{ln} = 0$ unless $l = 2k-1$, in order to change the summation from l to k.

To evaluate this sum we multiply the summands by x^k. The sum converges for $0 < x < 1$, and we shall see that the limit $x \to 1$ exists. We rewrite the sum as

$$\sum_{k=1}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k)(2k+n-1)(2k+m-2)(2k+m)} x^k =$$

$$\sum_{k=1}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k)} x^k \left(\frac{a}{2k+n-1} + \frac{b}{2k+m-2} + \frac{c}{2k+m}\right), \quad \text{(A.9)}$$

where

$$a = \frac{1}{m^2-2mn+n^2-1}\,, \quad b = \frac{1}{2-2m+2n}\,, \quad c = \frac{1}{2+2m-2n}\,. \quad \text{(A.10)}$$

For $x < 1$ we can change the summation order and sum each term separately. All three sums are of the same form. A general sum of this form gives

$$\sum_{k=1}^{\infty} \frac{\Gamma(\frac{1}{2}+k)x^k}{\Gamma(k)(k+u)} =$$

$$-\frac{\sqrt{\pi}((2u(x-1)+x-2)B_x(u+1,\frac{1}{2}) + (3+2u)B_x(u+1,\frac{3}{2}))}{2(x-1)x^u}\,, \quad \text{(A.11)}$$

where $B_x(u+1,\frac{1}{2})$ is the incomplete beta function. Expanding this term using $x = 1-\epsilon$ gives

$$-\sqrt{\frac{\pi}{\epsilon}} - \frac{\pi\Gamma(u+1)}{\Gamma(u+\frac{1}{2})} + O(\epsilon)\,. \quad \text{(A.12)}$$

Now we substitute

$$u = \frac{n-1}{2},\ \frac{m-2}{2},\ \frac{m}{2}\,, \tag{A.13}$$

multiply by a, b, c, and sum the three terms. The singular terms drop out since

$$a + b + c = 0, \tag{A.14}$$

and we are left with

$$-\frac{\pi}{2}\left(\frac{\Gamma(\frac{n+1}{2})}{((m-n)^2-1)\Gamma(\frac{n}{2})} - \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m+1}{2})}\frac{m+n-1}{4((m-n)^2-1)}\right). \tag{A.15}$$

Plugging this result into eq. (A.8) we get two terms which are symmetric with respect to interchanging m, n. This completes the proof that $[V^B, B] = 0$. Since all T_n are given functions of B we also get

$$[V^B, T_n] = 0\,. \tag{A.16}$$

A.2 The eigenvalues of the butterfly

After showing that the eigenvectors of V^B are the same as those of B, we turn to find the corresponding eigenvalues. All the twist even eigenvectors have a zero eigenvalue (A.5). For the twist odd eigenvectors it is enough to calculate

$$\lambda_{v_-^{(\kappa)}} = \frac{1}{v_{-,m}^{(\kappa)}}\sum_{n=1}^{\infty} V_{mn} v_{-,n}^{(\kappa)}\,, \tag{A.17}$$

for any m such that $v_{-,m}^{(\kappa)} \neq 0$. We choose to perform the calculation with $m = 1$. $v_-^{(\kappa)}$ is given by the generating function [**?**]

$$\sum_{m=1}^{\infty} \frac{v_{-,m}^{(\kappa)}}{\sqrt{m}} z^m = \frac{1}{\kappa}\sinh(\kappa \tan^{-1}(z))\,, \tag{A.18}$$

with the inverse relation

$$v_{-,m}^{(\kappa)} = \frac{\sqrt{m}}{2\pi i\kappa}\oint \frac{\sinh(\kappa\tan^{-1}(z))}{z^{m+1}} dz\,, \tag{A.19}$$

where z is a small contour around the origin.

We can now calculate the eigenvalues of eq. (A.17) setting $m = 1$

$$\begin{aligned}\lambda_{v^{(\kappa)}} &= \frac{1}{v^{(\kappa)}_{-,1}} \sum_{k=1}^{\infty} (-1)^k \frac{\sqrt{2k-1}}{2k} \frac{\Gamma(\frac{2k-1}{2})}{\sqrt{\pi}\Gamma(k)} v^{(\kappa)}_{-,2k-1} \\ &= \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k+1)} \oint \frac{\sinh(\kappa \tan^{-1}(z))}{2\pi i \kappa z^{2k}} dz\,,\end{aligned} \tag{A.20}$$

where in the first equality we set $n = 2k - 1$, and use the expression for the matrix V^B (A.1). In the second equality we use the generating function (A.19).

We would like to change the order of summation and integration. For that we need $|z| > 1$, but the generating function has two branch cuts, going from i to $i\infty$, and from $-i$ to $-i\infty$. Therefore we deform the contour as in figure 2. The contour is now given by the integrals on the small semi-circles with radius ϵ (marked A in the figure), the large semi-circles with radius R (B), and the straight lines around the cuts ($\mathrm{C}^{\pm}$). The contribution to the integral of the parts A and B goes to zero as $\epsilon \to 0$ and $R \to \infty$. Both integrals on C^+ contribute the same, and the same goes for both C^- integrals. We can now write the integral as

$$\begin{aligned}&\oint \frac{\sinh(\kappa \tan^{-1}(z))}{2\pi i \kappa z^{2k}} dz = 2 \lim_{\epsilon\to 0} \lim_{R\to\infty} \int_{\mathrm{C}^+ + \mathrm{C}^-} \frac{\sinh(\kappa \tan^{-1}(z))}{2\pi i \kappa z^{2k}} dz = \\ &\lim_{\epsilon\to 0^+} \int_1^{\infty} \frac{\sinh(\kappa \tan^{-1}(ix-\epsilon)) - \sinh(\kappa \tan^{-1}(ix+\epsilon))}{\pi\kappa (ix)^{2k}} dx = \\ &\frac{(-1)^k}{\pi\kappa} \int_1^{\infty} \frac{\sinh(\kappa(\frac{\pi}{2} + i \coth^{-1}(x))) - \sinh(\kappa(-\frac{\pi}{2} + i\coth^{-1}(x)))}{x^{2k}} dx = \\ &\frac{2(-1)^k \sinh\left(\frac{\kappa\pi}{2}\right)}{\pi\kappa} \int_1^{\infty} \frac{\cos(\kappa \coth^{-1}(x))}{x^{2k}} dx\,.\end{aligned} \tag{A.21}$$

Now we can change the order of summation and integration in eq. (A.20)

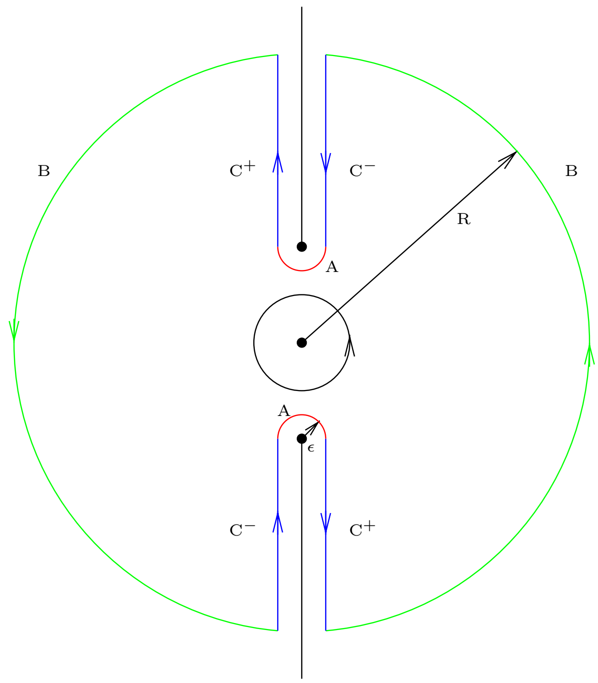

Figure 2: The original contour of integration (the small black circle) is replaced by the more complicated path that satisfies $|z| > 1$ except for the segment A. in a way that avoids the branch cuts. Only the blue lines (C$^\pm$) contribute in the limit $R \to \infty, \epsilon \to 0$.

to get

$$
\begin{aligned}
\lambda_{v_-^{(\kappa)}} &= \frac{2\sinh(\frac{\kappa\pi}{2})}{\pi^{3/2}\kappa} \int_1^\infty \cos(\kappa \coth^{-1}(x)) \sum_{k=1}^\infty \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)x^{2k}} dx \\
&= \frac{2\sinh(\frac{\kappa\pi}{2})}{\pi\kappa} \int_1^\infty \cos(\kappa \coth^{-1}(x))(\frac{x}{\sqrt{x^2-1}} - 1) dx \\
&= \frac{2\sinh(\frac{\kappa\pi}{2})}{\pi\kappa} \int_0^\infty \frac{\cos(\kappa u)}{1+\cosh(u)} du = \frac{1}{\cosh(\frac{\kappa\pi}{2})} \,.
\end{aligned} \tag{A.22}
$$

These are the twist odd eigenvalues of the butterfly. This result is in agreement with the results of the main text.